University of Illinois at Urbana-Champaign
Dept. of Electrical and Computer Engineering

# ECE 120: Introduction to Computing
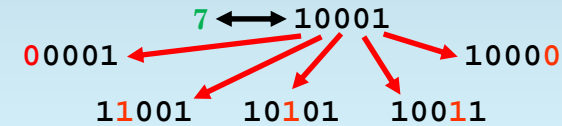
## Parity and Hamming Distance

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## A 2-out-of-5 Code Can Detect Any Single Bit Flip

A 2-out-of-5 code maps decimal digits
into 5-bit code words.

Each code word has exactly two 1 bits.

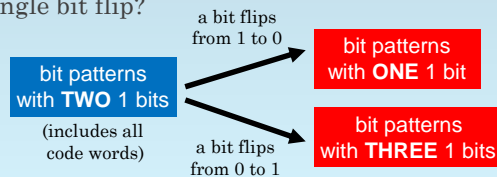How many 1 bits are there if we flip a bit?

7 ⟷ 10001

00001          10000

11001   10101   10011

Result: **Either ONE 1 bit or THREE 1 bits.**

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## Can We Generalize This Approach to Error Detection?

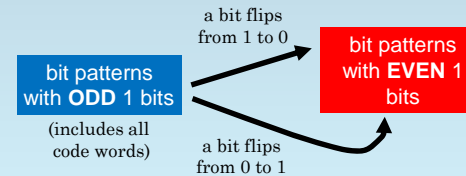A 2-out-of-5 code represents decimal digits.

What about binary numbers? Letters? Colors?

Is there a general strategy for handling a
single bit flip?

bit patterns
with **TWO** 1 bits
(includes all
code words)

a bit flips
from 1 to 0 → bit patterns
with **ONE** 1 bit

a bit flips
from 0 to 1 → bit patterns
with **THREE** 1 bits

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## Generalize by Using Even and Odd Numbers of 1 Bits

What if we choose all code words with
an ODD number of 1 bits?

bit patterns
with **ODD** 1 bits
(includes all
code words)

a bit flips
from 1 to 0 → bit patterns
with **EVEN** 1
bits

a bit flips
from 0 to 1

Result: **Any bit flip gives a non-code-word!**

## Add a Parity Bit to Any Representation!

starting with any representation
◦ Add one extra **parity bit** to each code word.
◦ Choose parity bit's value to make total number of 1 bits ODD (called odd parity).

For example, 3-bit unsigned with odd parity…

| | | | |
|---|---|---|---|
| **0** ⟷ `0001` | | **4** ⟷ `1000` |
| **1** ⟷ `0010` | | **5** ⟷ `1011` |
| **2** ⟷ `0100` | | **6** ⟷ `1101` |
| **3** ⟷ `0111` | | **7** ⟷ `1110` |

## Hamming Distance: The Number of Bits that Differ

Let's define a way to measure distance
◦ between two bit patterns
◦ as the number of bits that must change/flip

```
0101            0101
    DISTANCE 1      DISTANCE 3
0100            0010
```

We call this measure **Hamming distance**
(after Richard Hamming, a UIUC alumnus).

## Define the Hamming Distance for a Representation

Let's also define the Hamming distance for a representation (let's call a representation a **code** now):

◦ Given the set of code words (bit patterns) that have meaning,

◦ the Hamming distance of the code
  ◦ is the **minimum Hamming distance**
  ◦ between **any two distinct code words**.

## Example: The Hamming Distance of BCD is 1

What is the Hamming distance (H.D.) of BCD?
◦ Choose two code words,
◦ say those representing digits **0** and **1**.

```
0 ⟷ 0000
         DISTANCE 1
1 ⟷ 0001
```

H.D. of BCD is min. over all code word pairs.

**Thus BCD has Hamming distance 1.**

## Example: The Hamming Distance of 2-out-of-5 is 2

What is the H.D. of a 2-out-of-5 code?
- Choose two distinct code words, **A** and **B**.
- Each has two 1s (cannot be the same two).
- So **A** must have at least one 1 in a position where **B** has a 0.
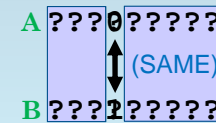- And **B** must have at least one 1 in a position where **A** has a 0.

H.D. from **A** to **B** is thus at least 2.

**Thus a 2-out-of-5 code has H.D. 2.**

## Example: H.D. with Odd Parity

What is the H.D. of a code with odd parity?
- Choose two distinct code words, **A** and **B**.
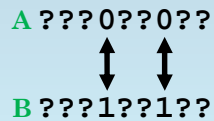
**A** `???0?????`
(SAME)
**B** `???1?????`

**A** must differ from **B** in some location.

- Assume that the location is unique.
- A has odd parity, so B has even parity (contradiction, so location cannot be unique).
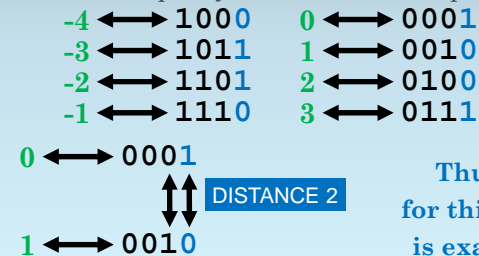
## Example: H.D. with Odd Parity is at Least 2

There must be **at least two** locations in which **A** to **B** differ.

**A** `???0??0??`

**B** `???1??1??`

**Thus H.D. with odd parity is at least 2.**

## H.D. of 2's Complement with Odd Parity is 2

Add an odd parity bit to 3-bit 2's complement:

| -4 ⟷ `1000` | 0 ⟷ `0001` |
|---|---|
| -3 ⟷ `1011` | 1 ⟷ `0010` |
| -2 ⟷ `1101` | 2 ⟷ `0100` |
| -1 ⟷ `1110` | 3 ⟷ `0111` |

0 ⟷ `0001`

DISTANCE 2

1 ⟷ `0010`

**Thus H.D. for this code is exactly 2.**

## With H.D. 2, Two-Bit Errors Can be Undetectable

What happens if two bit errors occur

when using 3-bit 2's complement with parity?

0 ⟷ 0001

first bit flip

0011

second bit flip                 In this case,
                                 no error can
0010 ⟷ 1                         be detected!

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## A Code with H.D. of d Allows Detection of (d-1) Errors

More generally…
◦ Start with a code with H.D. given by **d**
◦ Ask: How many errors can be detected?

H.D. of **d** implies
◦ Any code word is at least **d** flips from any other.
  ◦ Thus (**d**-1) bit errors cannot transform
    any code word into any other.
  ◦ Thus up to (**d**-1) bit errors can be detected.
◦ There exist code words **A** and **B** separated by
  **d** flips, so **d** bit errors can transform **A** into **B**.

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